ESC195

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1 11.1 - Sequences

The limit of a sequence $\{a_n\}$ is simply the limit of of the expression for the nth term with respect to x:

$$\lim_{x\to\infty} a_n = L$$

Theorem: If $\lim_{x\to\infty} f(x) = L$ and $f(n) = a_n$ when n is an integer, then $\lim_{n\to\infty} a_n = L$

The theorem states that if a function $f(x)$ approaches a limit L as x goes to infinity, and if the values of f(n) are represented by the sequence a_n for integer values of n, then the limit of the sequence a_n as n goes to infinity is also L. In other words, the limit of the function and the limit of the sequence are the same when x and n approach infinity, respectively.

Limit Laws for Sequences:
Without copying them here, all the laws are the same as for regular limits.

Power Law:

$$\lim_{x\to\infty} (a_n)^p = \left[\lim_{x\to\infty} a_n\right]^p \quad \textbf{if} \quad p > 0 \quad \textbf{AND} \quad a_n > 0$$

Absolute Value Convergence Theorem:

$$\text{If} \quad \lim_{x\to\infty} |a_n| = 0 \quad \text{then} \quad \lim_{x\to\infty} a_n = 0$$

Theorem:

If $\lim\limits_{x\to\infty} =$ L and the function f is continuous at L, then

$$\lim_{x\to\infty} f(a_n) = f(L)$$

If we apply a continuous function to the terms of a convergent sequence, the result is also convergent.

The sequence $\{r^n\}$ is convergent if $-1 \leq r \leq 1$ and divergent for all other values of r.

Definition: A sequence $\{a_n\}$ is called **increasing** if $a_n < a_{n+1}$ for all $n \geq 1$. i.e.: $a_1 < a_2 < a_3 \ldots$. It is called **decreasing** if the converse is true.

A sequence is called **monotonic** if it is *either increasing or decreasing*.

Monotonic Sequence Theorem: every bounded, monotonic sequence is convergent.

In particular, a sequence that is increasing and bounded above, converges, a sequence that is decreasing and bounded below converges.

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Sum of Geometric Series:
The sum of the geometric series:

$$\sum_{i=1}^{\infty} ar^{n-1} = a + ar + ar^2$$

and its sum is:

$$\frac{a}{1-r} \quad \text{for} \quad |r| < 1$$

Convergence Theorem 6: If a series is convergent, then the limit of the nth term approaches 0.

Test For Divergence: if $\lim_{x\to\infty} a_n$ does not exist or does not go to 0, then the series is divergent.

Series Splitting: A constant, c, can be moved into our out of the sum operator.

Additionally, two additive parts of series can be decompiled into the addition of separate series, or subtracted:

$$\sum(a_n + b_n) = \sum a_n + \sum b_n$$

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Integral Test: Suppose f is a continuous, positive, decreasing function on $[1, \infty]$ and let $a_n = f(n)$.

If $\int_1^\infty f(x)dx$ converges, then $\sum_{i=1}^{\infty} a_n$ converges.

If $\int_1^\infty f(x)dx$ diverges, then $\sum_{i=1}^{\infty} a_n$ diverges.

The p series:

$$\sum_{i=1}^{\infty} \frac{1}{n^p}$$

is convergent if $p > 1$, and divergent if $p \leq 1$

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Direct Comparison Test: Suppose $\sum a_n, \sum b_n$ are series with positive terms.

i) If $\sum b_n$ is convergent, and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

ii) Converse for divergent, $a_n \geq b_n$, divergent.

Limit Comparison Test: Given two series with positive terms: $\sum a_n, \sum b_n$:

$$\lim_{x\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either **both series converge, or both series diverge**.

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Alternating Series Test: If the alternating series:

$$\sum_{i=1}^{\infty}(-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + \dots$$

satisfies:

i) $b_{n+1} \leq b_n$ for all n

ii) $\lim_{x\to\infty} b_= 0$

Absolute Convergence:

The series $\sum a_n$ is called **absolutely convergent** if the series of absolute values: $\sum |a_n|$ is convergent.

Conditional Convergence:

A series $\sum a_n$ **conditionally convergent** if it is convergent but not absolutely convergent.

Theorem: If a series is absolutely convergent, then it is convergent.

6 11.6

Ratio Test:

i) if $\lim_{x\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \mathbf{L < 1}$, then the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent**.

ii) if $\lim_{x\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \mathbf{L > 1}$, then the series $\sum_{n=1}^{\infty} a_n$ is **divergent**.

iii) if $\lim_{x\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \mathbf{L = 1}$, then the **ratio test is inconclusive**.

Root Test:

i) if $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L < 1$ then the series is **absolutely convergent**.

ii) if $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L > 1$ then the series is **divergent**.

ii) if $\lim_{n\to\infty}\sqrt[n]{|a_n|} = L = 1$ then the test is **inconclusive**.

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Stratagies, no useful summary.

8 11.8

Power Series Convergence:

For $\sum_{n=0}^{\infty} c_n(x-a)^n$, there are only three possibilities

i) The series **only converges when $\mathbf{x = a}$**.

ii) The series **converges for all x**.

iii) There is a positive number r such that **the series converges if $|\mathbf{x - a}| < \mathbf{R}$** and **diverges if $|\mathbf{x - a > R}|$**.

The number **R, is the radius of convergence**.

9 11.9

Representation 1:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 \dots x^n \quad = \sum_{n=0}^{\infty} x^n \quad | \ x < 1$$

Theorem: If the power series $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function f defined by:

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 \cdots = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable and therefore continuous on the interval $(a-R, a+R)$

$$f'(x) = c_1 + 2c_2(x-1+3c_3(x-a)^2) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

...similar for integration.

10 10.10

Theorem: If f has a power series representation (expansion) at a, that is, if:

$$f(x) = \sum_{n=0}^{n} c_n(x-a)^n \quad |x-a| < R$$

then its coefficients are given by:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Figure 1:

Essentially:

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots$$

Maclaurin Series: Essentially the resulting expansion when $\mathbf{a = 0}$:

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 \ldots$$

Theorem: If $f(x) = T_n(x) + R_n(x)$, where T_n is the nth-degree Taylor polynomial of f at a, and if:

$$\lim_{n\to\infty} R_n(x) = 0$$

for $|x-a|<R$, then f is equal to the sum of its Taylor series on the interval $|x-a|<R$.

In other words, a function is the sum of its Taylor series if the limit of the remainder approaches 0.

Useful Limit:

$$\lim_{n\to\infty} \frac{x^n}{n!} = 0$$

for every real number x.

Maclaurin Series of e^x:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \qquad \text{for all } x$$

Binomial Series: **REVIEW IF WE HAVE TIME!**

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Parametric Line 1:

$$\mathbf{r} = {}_{\mathbf{0}} + t\mathbf{v}$$

Where: $\mathbf{r_0}$ is any point on the line.

$\mathbf{v}$ is the "slope" or "direction" of the line.

t is the parameter, that when varied results in the line structure.

Parametric Line 2: Parametric equations for a line through the point (x_0, y_0, z_0) and parallel to the direction vector $\langle a, b, c \rangle$ are:

$$x = x_0 + at \qquad y = y_0 + by \qquad z = z_0 + ct$$

Parametric Line 3: We can also eliminate the parameter and rearrange to get:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Parametric Line Segment: The line segment from $\mathbf{r_0}$ to $\mathbf{r_1}$ is given by the vector equation:

$$\mathbf{r}(t) = (1 - t)\mathbf{r_0} + t\mathbf{r_1} \quad 0 \leq t \leq 1$$

Plane Equation 1:

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r_0}) = 0$$

Where: $\mathbf{r}$ = arbitrary position on the plane, can be varied $\mathbf{r_0}$ = is a particular point on the plane $\mathbf{n}$ = vector normal to the plane

Alternatively:

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}$$

Scalar Equation of a Plane: through a point $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{nn} = \langle a, b, c \rangle$ is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Distance Formula: from a point $P_1(x_1, y_1, z_1)$ to the plane $ax+by+cz+d=0$ is:

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Cross Product: If: $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **cross product** of $\mathbf{a}$ and $\mathbf{b}$ is the vector:

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle$$

Which can be memorized as the sequence: **23, 31, 12**, who are prime, prime, and very divisible respectively. Also sum to 5, 4, and 3 respectively. And whose digits ascend, descend, and ascent, respectively.

12 12.6

Traces: Setting one variable equal to zero, like:

$$x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$$

Set $z = 0$:

$$x^2 + \frac{y^2}{9} = 1$$

We can easily recognize this as an ellipse, and extend the logic for substituting x or y to be 0, which would all be ellipses, since they're all ellipses, we have an ellipsoid.

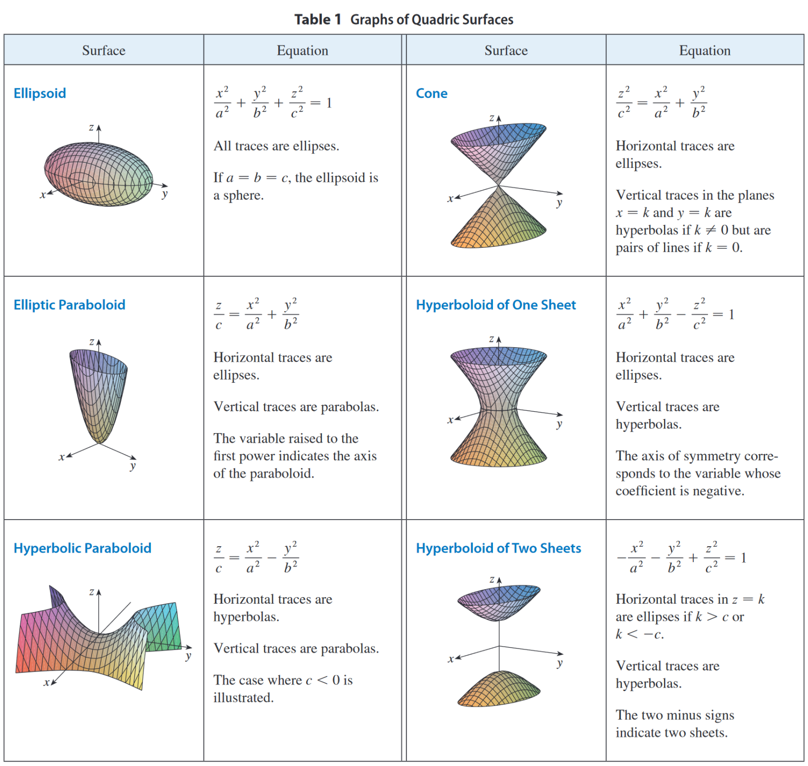

Table 1 Graphs of Quadric Surfaces

Surface	Equation	Surface	Equation
Ellipsoid	$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ All traces are ellipses. If $a = b = c$, the ellipsoid is a sphere.	Cone	$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces in the planes $x = k$ and $y = k$ are hyperbolas if $k \neq 0$ but are pairs of lines if $k = 0$.
Elliptic Paraboloid	$\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ Horizontal traces are ellipses. Vertical traces are parabolas. The variable raised to the first power indicates the axis of the paraboloid.	Hyperboloid of One Sheet	$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ Horizontal traces are ellipses. Vertical traces are hyperbolas. The axis of symmetry corresponds to the variable whose coefficient is negative.
Hyperbolic Paraboloid	$\frac{z}{c} = \frac{x^2}{a^2} - \frac{y^2}{b^2}$ Horizontal traces are hyperbolas. Vertical traces are parabolas. The case where $c < 0$ is illustrated.	Hyperboloid of Two Sheets	$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Horizontal traces in $z = k$ are ellipses if $k > c$ or $k < -c$. Vertical traces are hyperbolas. The two minus signs indicate two sheets.

Figure 2:

13 13.3

Arc Length of Space Curves:

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} dt$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$

Re-parameterization of Arc Length Suppose we want to reparameterize from the initial point $(1, 0, 0)$

$$\mathbf{r}(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{i} + t\mathbf{k}$$

...

$$\text{arc length} = s = \sqrt{2}t$$

Then substitute $t = \frac{s}{\sqrt{2}}$...

Unit Tangent Vector:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Curvature: The **curvature** of a curve is:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}/dt}{ds/dt}\right| = \frac{|\mathbf{T}'(t)|}{\mathbf{r}'(t)}$$

Curvature Expression 2:

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}]$$

Which can be memorized as the magnitude of the cross product of the first derivative, second derivative, and divided by magnitude the first derivative 3 times.

Curvature of f(x):

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}}$$

Which can be remembered as absolute value of second derivative divided by the inside of the arc length formula, but instead of square rooting, we square root and then raise to 3!

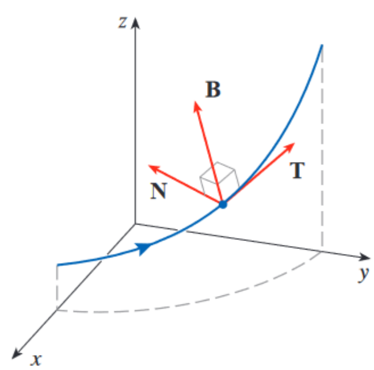

Figure 3: Normal, Tangent, and Binormal Vectors

Normal Vector of Space Curve:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

Binormal Vector of Space Curve:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

Torsion: REVIEW IF WE HAVE TIME

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Acceleration Vector?

$$\mathbf{a} = v'\mathbf{T} + \kappa v^2 \mathbf{N}$$

15 14.3

Partial Derivative notation: if $z = f(x, y)$, we write:

$$f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_1 = D_1 f = d_x f$$

...similar for y

Rule for finding partial derivatives of $\mathbf{z = f(x, y)}$:

1. To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x.
2. To find f_y Do the exact same, but treat x as a constant, and differentiate with respect to y.

Fourier Series: **ChadGPT Summary:** (I need to sleep -15 minutes ago):

1. **The main idea:** Any periodic function (a function that repeats itself) can be represented as a sum of sines and cosines with different frequencies and amplitudes.

2. **The formula:** For a function $f(x)$ with period T, the Fourier series is given by:

$$f(x) \approx a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2\pi nx}{T}\right) + b_n \sin\left(\frac{2\pi nx}{T}\right) \right]$$

   where n ranges from 1 to ∞, and a_0, a_n, and b_n are the Fourier coefficients.

3. **How to find the coefficients:**

$$a_0 = \frac{1}{T} \int_0^T f(x)\, dx$$

$$a_n = \frac{2}{T} \int_0^T f(x) \cos\left(\frac{2\pi nx}{T}\right)\, dx$$

$$b_n = \frac{2}{T} \int_0^T f(x) \sin\left(\frac{2\pi nx}{T}\right)\, dx$$

4. **Steps to find the Fourier series representation:**

   (a) Determine if your function is periodic, and find its period T.

   (b) Calculate the Fourier coefficients (a_0, a_n, and b_n) using the above formulas.

   (c) Plug the coefficients into the Fourier series formula to get the representation.